3d Gauge Theories, Symplectic Duality and Knot Homology IV

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Recall that we specified a 3d gauge theory using a group G acting on $V = (T^*\mathbb{C})^N$ by hyperkähler isometries. It has a commutant G_H. The Higgs branch is the hyperk"ahler quotient $V///G$, which is acted on by G_H, while the Coulomb branch M_C is another branch of the moduli space of vacua acted on by another group G_C which includes at least a copy of $U(1)$ for every copy of $U(1)$ in G.

To get a statement about symplectic duality we need extra ingredients. Symplectic duality is about categories of things, e.g. D-modules, and these arise when studying boundary conditions.

1. We can formulate the theory on $\mathbb{R}^2 \times \mathbb{R}_+$ and then we can ask for boundary conditions at $0 \in \mathbb{R}_+$. We'll ask for boundary conditions that preserve half of the supersymmetries. (We can't preserve all of them because that would imply translation invariance, which we've killed.) The possible boundary conditions come from possible supersymmetry algebras in 2 dimensions. We'll ask for $(2,2)$ SUSY. Attached to M_C and M_H are boundary conditions B_C, B_H supported on holomorphic Lagrangian submanifolds (this comes from $(2,2)$ SUSY).

2. We need a topological twist specified by a supersymmetry Q with $Q^2 = 0$ with respect to which we'll take Q-cohomology.

3. We need a twist symmetric under exchanging M_H and M_C.

Recall from Kevin's talk that there are three twists available in 3d $N = 4$ gauge theory. One of them is a Rozansky-Witten twist such that the algebra of local operators becomes the chiral ring $\mathbb{C}[M_C]$ of the Coulomb branch M_C. Another is a Rozansky-Witten twist such that the algebra of local operators becomes the chiral ring $\mathbb{C}[M_H]$. But these are not symmetric under exchanging the Coulomb and Higgs branches. There is a holomorphic twist which does have this symmetry but which is not topological. But it is topological after compactifying to 2d.

Recall that when scaling the m, t parameters via $m, t \mapsto \lambda m, \lambda^{-1} t$, then as $\lambda \to \infty$ we get a sigma model with target M_C and as $\lambda \to 0$ we get a sigma model with target M_H. We can now compactify both of these on a circle S^1 of radius R. Here things get tricky.

In the compactification we want to do another scaling such that the product Rmt remains constant. We will perform the scaling

$$(R, m, t) \mapsto (\eta R, \eta^{-1/2} m, \eta^{-1/2} t). \tag{1}$$

and take $\eta \to 0$. This gets us very close to a 2d M_H or M_C sigma model. The effective mass on the two sides is different: in the 2d M_H model we have effective mass $\eta^{-1/2} m$ and volume $\eta^{1/2} tR$, and in the 2d M_C model we have effective mass $\eta^{-1/2} t$ and volume $\eta^{1/2} mR$.

Q: what does mt mean? (They live in the Cartan subalgebras $t \in \mathfrak{h}_{G_C}$ and $m \in \mathfrak{h}_{G_H}$ of two different Lie algebras.)

A: formally it's the value of a real moment map in a vacuum, which is bilinear in m and t.

If we make λ extremely close to 0 and make η kind of small, then the 2d sigma model into M_H will be very massive and have very small volume. Similarly for M_C.

Conjecture 0.1. *The 2d equivariant B-model categories (together with a scaling limit projecting it to a zero-graded sector) become category O (in the sense of the first talk) O_{M_C} for the Coulomb branch and category O O_{M_H} for the Higgs branch respectively. These are formally equivalent to categories of D-modules.*

As $\eta \to 0$, we get a massive Landau-Ginzburg model with target the Cartan subalgebra of G. One way to derive this is to use Hori-Vafa mirror symmetry.

Conjecture 0.2. *Both O_{M_C} and O_{M_H} are equivalent to the Landau-Ginzburg category, or equivalently the A-model Fukaya-Seidel category.*

To define the categories O above we need to quantize the chiral rings $\mathbb{C}[M_C], \mathbb{C}[M_H]$. We'll do this by splitting $\mathbb{R}^3$ into $\mathbb{R}^2 \times \mathbb{R}$. Let $V = \frac{\partial}{\partial phi}$ generate rotation in the first variable. If R_C, R_H are the Rozansky-Witten supercharges we want to alter their squares to

$$Q_C^2 = \hbar L_V, Q_H^2 = \hbar L_V. \tag{2}$$

This introduces a noncommutativity into the operator product expansion. The resulting deformation is essentially unique up to a dependence on the parameters $m^{\mathbb{C}}$ and $t^{\mathbb{C}}$. These define classes in $H^2(M_C, \mathbb{C})$ and $H^2(M_H, \mathbb{C})$ respectively. Integral values of these classes are special: setting $m^{\mathbb{C}}, t^{\mathbb{C}} \in \hbar\mathbb{Z}$ is classically like setting them to zero.

Example When $G = U(1)$ we have $M_H = T^*\mathbb{CP}^N$. The chiral ring gets quantized to a quotient of $U(_N)$ (reflecting the fact that we are not working on the entire flag variety) obtained by quantizing $\mathbb{C}\utilde{[X_i, Y_i}]/([X_i, Y_i] = \hbar\delta_{ij})$ and pass to a symplectic quotient.

We also have $M_C = \mathbb{C}^2/\mathbb{Z}_N$. The chiral ring, which used to be defined by a relation

$$V_+V_- = \prod(\varphi + m_i^{\mathbb{C}}) = P(\varphi) \tag{3}$$

gets quantized to

$$V_+V_- = P\left(\varphi - \frac{\hbar}{2}\right), V_-V_+ = P\left(\varphi + \frac{\hbar}{2}\right) \tag{4}$$

and

$$[\varphi, V_\pm] = \pm\hbar V_\pm. \tag{5}$$

These are related to Yangians and finite W-algebras.

A nice way to look at boundary conditions is to start with boundary conditions for the gauge theory and move them to either the Higgs or Coulomb branches. For starters, there are boundary conditions $B_\beta, \beta \in \{\pm 1\}^N$ which do the following thing. Everything is either

Neumann or Dirichlet boundary conditions: either the derivatives or the values gets fixed. In the vector multiplet we require

$$\partial_{\mathbb{R}_+} A_{\mathbb{R}^2} \mid_0 = 0, \partial_{\mathbb{R}_+} \varphi \mid_0 = 0, (\sigma + i\gamma) \mid_0 = t_{2d} \tag{6}$$

where t_{2d} is a 2d FI parameter which has nothing to do with our 3d FI parameter. In the hypermultiplet we require

$$\beta_i = +1 \Rightarrow \partial_{\mathbb{R}_+} X_i \mid_0 = Y_i \mid_0 = 0 \tag{7}$$

and

$$\beta_i = -1 \Rightarrow X_i \mid_0 = \partial_{R_+} Y_i \mid_0 = 0. \tag{8}$$

On the Higgs branch these boundary conditions B^H_β become smooth, irreducible holomorphic Lagrangians. Not all of the βs survive. The ones that do are supported on a nice part M^+_H of the Higgs branch. These become simples in O_{M_H}.

On the Coulomb branch we get holomorphic Lagrangians, but they have the wrong support unless we take $t_{2d} \to \pm\infty$, where they deform onto M^+_C. These become projectives in O_{M_C}.

(Several questions and answers happen here that I didn't catch.)